Four-Body Bound State Calculations in Three-Dimensional Approach

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Abstract

The four-body bound state with two-body interactions is formulated in Three-Dimensional approach, a recently developed momentum space representation which greatly simplifies the numerical calculations of few-body systems without performing the partial wave decomposition. The obtained three-dimensional Faddeev-Yakubovsky integral equations are solved with two-body spin-independent and spin-averaged potentials. This is the first step toward the calculations of four-nucleon bound state problem in Three-Dimensional approach. Results for four-body binding energies are in good agreement with achievements of the other methods.

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I. INTRODUCTION

The bound state of few-body systems seems to be an ideal laboratory to determine two-, three- and four-body nuclear forces. The studies of the four-body bound state properties for the case of few-body interactions have received increasing attention theoretically and experimentally in recent years. Although the four-body bound state poses a challenging problem numerically, because of presence of fourth body, its investigation promises insights into the rich structure of nuclear interactions. To this aim one requires an accurate and reliable method to obtain the full solution of four-body bound state in a straightforward manner.

The four-body bound state calculations are carried out by different methods to solve the nonrelativistic Schrödinger equation such as, the Coupled-Rearrangement-Channel Gaussian-basis Variational(CRCGV) [1]-[7], the Stochastic Variation(SV) with correlated Gaussians [8]-[12], the Hyperspherical Harmonic variational (HH) [13]-[18], the Green's Function Monte Carlo(GFMC) [19]-[22], the No-Core Shell Model(NCSM) [23]-[28], Effective Interaction Hyperspherical Harmonic(EIHH) [29],[30] and the Faddeev-Yakubovsky (F-Y). In the last method the nonrelativistic schrödinger equation is transformed to two coupled sets of finite number of coupled equations in three variables for the F-Y amplitudes. The calculations based on F-Y are performed in configuration space [31]-[34] and in momentum space [35]-[43] after a partial wave (PW) expansion, where the algebraic and algorithmic steps can be quiet involved. Though a few partial waves often provide qualitative insight, modern four-body calculations need 1572 or more different spin, isospin and angular momentum combinations [42],[43]. It appears therefore natural to avoid a PW representation completely and work directly with vector variables. On this basis in recent years W. Glöckle and collaborators have introduced the Three-Dimensional (3D) approach which greatly simplifies the two- and three-body scattering and bound state calculations without using PW decomposition [44]-[53].

Our aim in this paper is to extend this approach for four-body bound state with two-body interactions, we work directly with vector variables in the F-Y scheme in momentum space. Here the dependence on momentum vectors, i.e. the magnitudes of momenta and the angles between the momentum vectors, shows that the full solution can be reached exactly and simply whereas the PW representation of the amplitudes leads to rather complicated

expressions [54]. The calculations of four-body bound state with the three-body interactions is currently underway and it will be reported elsewhere. As a simplification we neglect spin and isospin degrees of freedom here and study the four-boson bound state. So this work is the first step in the direction of solving the four-nucleon bound state problem without performing the PW decomposition.

Recently the four-boson bound state has been studied with short-range forces and large scattering length at leading order in an Effective Field Theory approach [55]-[57], but this investigation is also based on PW decomposition and the interactions are restricted to only S-wave sector.

This paper is organized as follows. Section II reviews the F-Y equations for four-boson bound state. In section III we represent the coupled F-Y equations as function of momentum vectors. In section IV we discuss our choice for independent variables of momentum and angle variables for the unknown amplitudes in the equations and in their kernels, where this new representation(3D) is contrasted with traditional PW representation. Section V describes details of our algorithm for solving coupled F-Y three-dimensional integral equations. In section VI we compare our results for three- and four-boson binding energies to results obtained from other techniques. In order to test our calculation we investigate the stability of the eigenvalue of the Yakubovsky kernel with respect to the number of grid points and we calculate the expectation value of Hamiltonian operator. Finally we summarize in section VII and provide an outlook.

II.   FOUR-BODY BOUND STATE EQUATIONS

The bound state of four identical particles which interact via pairwise forces $V_{ij}(ij \equiv 12, 13, 14, 23, 24$ and $34)$ is given by Schrödinger equation which reads in integral form:

$$|\Psi\rangle = G_0 \sum_{i<j} V_{ij}|\Psi\rangle \tag{1}$$

Here the free four-body propagator is given by $G_0 = (E - H_0)^{-1}$, and H_0 stands for the free hamiltonian. Introducing Yakubovsky components $|\Psi\rangle = \sum |\psi_{ij}\rangle$, with $|\psi_{ij}\rangle = G_0 V_{ij}|\Psi\rangle$ leads to the six coupled integral equations:

$$|\psi_{ij}\rangle = G_0 t_{ij} \sum_{kl \neq ij} |\psi_{kl}\rangle \tag{2}$$

The operator t_{ij} describes the two-body $t-$matrix in the two-body subsystem ij. We can rewrite Eq. (2) as:

$$|\psi_{ij}\rangle = G_0 t_{ij}(|\psi_{ik}\rangle + |\psi_{il}\rangle + |\psi_{jk}\rangle + |\psi_{jl}\rangle + |\psi_{kl}\rangle) \tag{3}$$

Among various possibilities to decompose $|\psi_{ij}\rangle$ into three F-Y components we choose the following one:

$$|\psi_{ijk,l;ij}\rangle = G_0 t_{ij}(|\psi_{ik}\rangle + |\psi_{jk}\rangle)$$

$$|\psi_{ijl,k;ij}\rangle = G_0 t_{ij}(|\psi_{il}\rangle + |\psi_{jl}\rangle)$$

$$|\psi_{ij,kl;ij}\rangle = G_0 t_{ij}|\psi_{kl}\rangle \tag{4}$$

The F-Y component $|\psi_{ijk,l;ij}\rangle$ $(|\psi_{ij,kl;ij}\rangle)$ belongs to a $3+1$ $(2+2)$ partition. They fulfill the following relation:

$$|\psi_{ij}\rangle = |\psi_{ijk,l;ij}\rangle + |\psi_{ijl,k;ij}\rangle + |\psi_{ij,kl;ij}\rangle \tag{5}$$

The multiple indices for the F-Y components denote the two-body followed by the $3+1$ or $2+2$ fragmentation. It is easily seen that every $|\psi_{ij}\rangle$ component contains two $3+1$ type chains and one $2+2$ type chain, therefore total wave function $|\Psi\rangle$ contains twelve different $3+1$ type chains and six $2+2$ type chains. So altogether one has eighteen F-Y components. If we consider identical particles (here bosons, since we are omitting spin), the four-body wave function $|\Psi\rangle$ has to be totally symmetric. As a consequence all twelve components of $3+1$ type are identical in their functional form and only the particles are permuted. The same is true for the six components of $2+2$ type. Thus it is sufficient to consider only two independent F-Y components corresponding to the $3+1$ and $2+2$ partitions,

$$|\psi_1\rangle = |\psi_{123,4;12}\rangle$$

$$|\psi_2\rangle = |\psi_{12,34;12}\rangle \tag{6}$$

After the straightforward derivation the 18 coupled F-Y components shrink to two coupled F-Y equations:

$$|\psi_1\rangle = G_0 t_{12} P[(1 + P_{34})|\psi_1\rangle + |\psi_2\rangle]$$

$$|\psi_2\rangle = G_0 t_{12} \tilde{P}[(1 + P_{34})|\psi_1\rangle + |\psi_2\rangle] \tag{7}$$

Where P_{ij} is the permutation operator between the $i-th$ and $j-th$ particle, and

$$P = P_{12}P_{23} + P_{13}P_{23}$$

$$\tilde{P} = P_{13}P_{24} \tag{8}$$

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The total four-body wave function is then given as:

$$|\Psi\rangle = (1 + P + P_{34}P + \tilde{P})[(1 + P_{34})|\psi_1\rangle + |\psi_2\rangle] \tag{9}$$

The symmetry property of $|\psi_1\rangle$ under exchange of particles 1 and 2, and $|\psi_2\rangle$ under separate exchanges of particles $1, 2$ and $3, 4$ guarantee that $|\Psi\rangle$ is totally symmetric.

We would like to add the remark that another derivation of F-Y components is also possible [35]. In this representation two transition operators which follow the subcluster Faddeev-like equations have been introduced as a function of two-body transition operator t_{12}. Consequently the kernel of coupled Yakubovsky integral equations contains two subcluster kernels that should be evaluated by Padé technique. So its numerical calculations is more complicated and time consuming in comparison to above derivation.

III. MOMENTUM SPACE REPRESENTATION OF FADDEEV-YAKUBOVSKY EQUATIONS

In order to solve the coupled equations (7), in momentum space we introduce standard Jacobi momenta sets corresponding to both $3 + 1\,(123, 4; 12)$ and $2 + 2\,(12, 34; 12)$ chains:

$$
\begin{aligned}
\vec{u}_1 &= \frac{\vec{k}_1 - \vec{k}_2}{2} \\
\vec{u}_2 &= \frac{2}{3}(\vec{k}_3 - \frac{\vec{k}_1 + \vec{k}_2}{2}) \\
\vec{u}_3 &= \frac{3}{4}(\vec{k}_4 - \frac{\vec{k}_1 + \vec{k}_2 + \vec{k}_3}{3}) \\[2mm]
\vec{v}_1 &= \frac{\vec{k}_1 - \vec{k}_2}{2} \\
\vec{v}_2 &= \frac{\vec{k}_1 + \vec{k}_2}{2} - \frac{\vec{k}_3 + \vec{k}_4}{2} \\
\vec{v}_3 &= \frac{\vec{k}_3 - \vec{k}_4}{2}
\end{aligned}
\tag{10}
$$

Then we introduce the four-body basis states corresponding to each Jacobi momenta set:

$$|\vec{u}_1\,\vec{u}_2\,\vec{u}_3\rangle$$

$$|\vec{v}_1\,\vec{v}_2\,\vec{v}_3\rangle \tag{11}$$

Both basis states are complete in the four-body Hilbert space:

$$\int D^3A \; |\vec{A}_1 \, \vec{A}_2 \, \vec{A}_3\rangle\langle\vec{A}_1 \, \vec{A}_2 \, \vec{A}_3| = 1 \qquad (12)$$

Where $\vec{A}_i$ indicates each one of $\vec{u}_i$ and $\vec{v}_i$ vectors and $D^3A \equiv d^3A_1 \, d^3A_2 \, d^3A_3$. Also they are normalized according to:

$$\langle\vec{A}_1 \, \vec{A}_2 \, \vec{A}_3|\vec{A}'_1 \, \vec{A}'_2 \, \vec{A}'_3\rangle = \delta^3(\vec{A}_1 - \vec{A}'_1) \, \delta^3(\vec{A}_2 - \vec{A}'_2) \, \delta^3(\vec{A}_3 - \vec{A}'_3) \qquad (13)$$

Clearly the basis states $|\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3\rangle$ are adequate to expand F-Y component $|\psi_1\rangle$ and correspondingly the basis states $|\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3\rangle$ are adequate for $|\psi_2\rangle$. Let us now represent coupled equations, Eq. (7), with respect to the basis states have been introduced in Eq. (11):

$$
\begin{aligned}
\langle\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3|\psi_1\rangle \;=\; & \int D^3u'' \, \langle\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3|G_0 tP(1+P_{34})|\vec{u}''_1 \, \vec{u}''_2 \, \vec{u}''_3\rangle\langle\vec{u}''_1 \, \vec{u}''_2 \, \vec{u}''_3|\psi_1\rangle \\
& + \int D^3v' \, \langle\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3|G_0 tP|\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3\rangle\langle\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3|\psi_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3|\psi_2\rangle \;=\; & \int D^3u' \, \langle\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3|G_0 t\tilde{P}(1+P_{34})|\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3\rangle\langle\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3|\psi_1\rangle \\
& + \int D^3v' \, \langle\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3|G_0 t\tilde{P}|\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3\rangle\langle\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3|\psi_2\rangle \qquad (14)
\end{aligned}
$$

It is convenient to insert again the completeness relations between permutation operators, it results:

$$
\begin{aligned}
\langle\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3|\psi_1\rangle \;=\; & \int D^3u' \int D^3u'' \, \langle\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3|G_0 tP|\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3\rangle \\
& \times\langle\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3|(1+P_{34})|\vec{u}''_1 \, \vec{u}''_2 \, \vec{u}''_3\rangle \, \langle\vec{u}''_1 \, \vec{u}''_2 \, \vec{u}''_3|\psi_1\rangle \\
& + \int D^3u' \int D^3v' \, \langle\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3|G_0 tP|\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3\rangle \\
& \times\langle\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3|\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3\rangle \, \langle\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3|\psi_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3|\psi_2\rangle \;=\; & \int D^3v' \int D^3u' \, \langle\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3|G_0 t\tilde{P}|\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3\rangle \\
& \times\langle\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3|(1+P_{34})|\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3\rangle\langle\vec{u}'_1 \, \vec{u}'_2 \, \vec{u}'_3|\psi_1\rangle \\
& + \int D^3v' \, \langle\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3|G_0 t\tilde{P}|\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3\rangle\langle\vec{v}'_1 \, \vec{v}'_2 \, \vec{v}'_3|\psi_2\rangle \qquad (15)
\end{aligned}
$$

For evaluating the coupled equations, Eq. (15), we need to evaluate the following matrix elements:

$$\langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | G_0 t P | \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' \rangle \tag{16}$$

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | G_0 t \tilde{P} | \vec{v}_1' \, \vec{v}_2' \, \vec{v}_3' \rangle \tag{17}$$

$$\langle \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' | (1 + P_{34}) | \vec{u}_1'' \, \vec{u}_2'' \, \vec{u}_3'' \rangle \tag{18}$$

$$\langle \vec{v}_1' \, \vec{v}_2' \, \vec{v}_3' | (1 + P_{34}) | \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' \rangle \tag{19}$$

For evaluating the first term, Eq. (16), we should insert again a completeness relation between the two-body $t-$matrix operator and permutation operator P as:

$$
\langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | G_0 t P | \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' \rangle = \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}}
$$
$$
\times \int D^3 u'' \langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | t | \vec{u}_1'' \, \vec{u}_2'' \, \vec{u}_3'' \rangle \langle \vec{u}_1'' \, \vec{u}_2'' \, \vec{u}_3'' | P | \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' \rangle \tag{20}
$$

Where the matrix elements of two-body $t-$matrix and permutation operator P are evaluated separately as:

$$
\langle \vec{u_1} \, \vec{u_2} \, \vec{u_3} | t | \vec{u}_1'' \, \vec{u}_2'' \, \vec{u}_3'' \rangle = \delta^3(\vec{u_2} - \vec{u}_2'') \, \delta^3(\vec{u_3} - \vec{u}_3'') \, \langle \vec{u_1} | t(\epsilon) | \vec{u}_1'' \rangle
$$
$$
\epsilon = E - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m} \tag{21}
$$

$$
\langle \vec{u}_1'' \, \vec{u}_2'' \, \vec{u}_3'' | P | \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' \rangle = \delta^3(\vec{u}_3'' - \vec{u}_3')
$$
$$
\times \ \{ \ \delta^3(\vec{u}_1'' + \frac{1}{2}\vec{u}_1' - \frac{3}{4}\vec{u}_2') \, \delta^3(\vec{u}_2'' + \vec{u}_1' + \frac{1}{2}\vec{u}_2')
$$
$$
+ \delta^3(\vec{u}_1'' + \frac{1}{2}\vec{u}_1' + \frac{3}{4}\vec{u}_2') \, \delta^3(\vec{u}_2'' - \vec{u}_1' + \frac{1}{2}\vec{u}_2') \ \} \tag{22}
$$

For evaluation the matrix elements of permutation operator P we have used the relation between Jacobi momenta in different two-body subsystems $(312, 4; 12), (231, 4; 12)$ and $(123, 4; 12)$. Inserting Eqs. (21) and (22) into Eq. (20) leads to:

$$
\langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | G_0 t P | \vec{u}_1' \, \vec{u}_2' \, \vec{u}_3' \rangle = \frac{\delta^3(\vec{u}_3 - \vec{u}_3')}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}}
$$
$$
\{ \ \delta^3(\vec{u}_2 + \vec{u}_1' + \frac{1}{2}\vec{u}_2') \, \langle \vec{u}_1 | t(\epsilon) | \frac{1}{2}\vec{u}_2 + \vec{u}_2' \rangle
$$
$$
+ \delta^3(\vec{u}_2 - \vec{u}_1' + \frac{1}{2}\vec{u}_2') \, \langle \vec{u}_1 | t(\epsilon) | \frac{-1}{2}\vec{u}_2 - \vec{u}_2' \rangle \} \tag{23}
$$

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Representation of the second term, Eq. (17), follows the similar steps:

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | G_0 t \tilde{P} | \vec{v}\,'_1 \, \vec{v}\,'_2 \, \vec{v}\,'_3 \rangle = \frac{1}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \int D^3 v'' \langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | t | \vec{v}\,''_1 \, \vec{v}\,''_2 \, \vec{v}\,''_3 \rangle$$
$$\times \; \langle \vec{v}\,''_1 \, \vec{v}\,''_2 \, \vec{v}\,''_3 | \tilde{P} | \vec{v}\,'_1 \, \vec{v}\,'_2 \, \vec{v}\,'_3 \rangle \tag{24}$$

The matrix elements of two-body $t-$matrix and permutation operator $\tilde{P}$ are evaluated as:

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | t | \vec{v}\,''_1 \, \vec{v}\,''_2 \, \vec{v}\,''_3 \rangle = \delta^3(\vec{v}_2 - \vec{v}\,''_2) \, \delta^3(\vec{v}_3 - \vec{v}\,''_3) \, \langle \vec{v}_1 | t(\epsilon^*) | \vec{v}\,''_1 \rangle$$
$$\epsilon^* = E - \frac{v_2^2}{2m} - \frac{v_3^2}{m} \tag{25}$$

$$\langle \vec{v}\,''_1 \, \vec{v}\,''_2 \, \vec{v}\,''_3 | \tilde{P} | \vec{v}\,'_1 \, \vec{v}\,'_2 \, \vec{v}\,'_3 \rangle = \delta^3(\vec{v}\,''_1 - \vec{v}\,'_3) \, \delta^3(\vec{v}\,''_2 + \vec{v}\,'_2) \, \delta^3(\vec{v}\,''_3 - \vec{v}\,'_1) \tag{26}$$

Inserting Eqs. (25) and (26) into Eq. (24) leads to:

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | G_0 t \tilde{P} | \vec{v}\,'_1 \, \vec{v}\,'_2 \, \vec{v}\,'_3 \rangle = \frac{\delta^3(\vec{v}_2 + \vec{v}\,'_2) \, \delta^3(\vec{v}_3 - \vec{v}\,'_1)}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \, \langle \vec{v}_1 | t(\epsilon^*) | \vec{v}\,'_3 \rangle \tag{27}$$

For evaluation the third term, Eq. (18), we should use the relation between Jacobi momenta in different chains $(123, 4; 12)$ and $(124, 3; 12)$, which leads to:

$$\langle \vec{u}\,'_1 \, \vec{u}\,'_2 \, \vec{u}\,'_3 | (1 + P_{34}) | \vec{u}\,''_1 \, \vec{u}\,''_2 \, \vec{u}\,''_3 \rangle = \delta^3(\vec{u}\,'_1 - \vec{u}\,''_1)$$
$$\times \; \{ \; \delta^3(\vec{u}\,'_2 - \vec{u}\,''_2) \, \delta^3(\vec{u}\,'_3 - \vec{u}\,''_3)$$
$$+ \delta^3(\vec{u}\,'_2 - \frac{1}{3}\vec{u}\,''_2 - \frac{8}{9}\vec{u}\,''_3) \, \delta^3(\vec{u}\,'_3 - \vec{u}\,''_2 + \frac{1}{3}\vec{u}\,''_3) \; \} \tag{28}$$

And finally for evaluation the fourth term, Eq. (19), we should use the relation between Jacobi momenta in two naturally different chains $(123, 4; 12)$ and $(12, 34; 12)$, which leads to:

$$\langle \vec{v}\,'_1 \, \vec{v}\,'_2 \, \vec{v}\,'_3 | (1 + P_{34}) | \vec{u}\,'_1 \, \vec{u}\,'_2 \, \vec{u}\,'_3 \rangle = \delta^3(\vec{u}\,'_1 - \vec{v}\,'_1)$$
$$\times \; \{ \; \delta^3(\vec{u}\,'_2 + \frac{2}{3}\vec{v}\,'_2 - \frac{2}{3}\vec{v}\,'_3) \, \delta^3(\vec{u}\,'_3 + \frac{1}{2}\vec{v}\,'_2 + \vec{v}\,'_3)$$
$$+ \delta^3(\vec{u}\,'_2 + \frac{2}{3}\vec{v}\,'_2 + \frac{2}{3}\vec{v}\,'_3) \, \delta^3(\vec{u}\,'_3 + \frac{1}{2}\vec{v}\,'_2 - \vec{v}\,'_3) \; \} \tag{29}$$

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Finally inserting Eqs. (23), (27), (28) and (29) in Eq. (15) yields:

$$\langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | \psi_1 \rangle \;=\; \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \int d^3 u_2' \, \langle \vec{u}_1 | t_s(\epsilon) | \frac{1}{2} \vec{u}_2 + \vec{u}_2' \rangle$$

$$\times \; \{ \; \langle \vec{u}_2 + \frac{1}{2} \vec{u}_2' \; \vec{u}_2' \; \vec{u}_3 | \psi_1 \rangle$$

$$+ \langle \vec{u}_2 + \frac{1}{2} \vec{u}_2' \; \frac{1}{3} \vec{u}_2' + \frac{8}{9} \vec{u}_3 \; \vec{u}_2' - \frac{1}{3} \vec{u}_3 | \psi_1 \rangle$$

$$+ \langle \vec{u}_2 + \frac{1}{2} \vec{u}_2' \; - \vec{u}_2' - \frac{2}{3} \vec{u}_3 \; \frac{1}{2} \vec{u}_2' - \frac{2}{3} \vec{u}_3 | \psi_2 \rangle \; \}$$

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \psi_2 \rangle \;=\; \frac{\frac{1}{2}}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \int d^3 v_3' \, \langle \vec{v}_1 | t_s(\epsilon^*) | \vec{v}_3' \rangle$$

$$\times \; \{ \; 2 \, \langle \vec{v}_3 \; \frac{2}{3} \vec{v}_2 + \frac{2}{3} \vec{v}_3' \; \frac{1}{2} \vec{v}_2 - \vec{v}_3' | \psi_1 \rangle$$

$$+ \langle \vec{v}_3 \; - \vec{v}_2 \; \vec{v}_3' | \psi_2 \rangle \; \} \tag{30}$$

Here $\langle \vec{a} | t_s(\varepsilon) | \vec{b} \rangle$ generally represents the symmetrized two-body $t-$matrix which is defined as,

$$\langle \vec{a} | t_s(\varepsilon) | \vec{b} \rangle \;=\; \langle \vec{a} | t(\varepsilon) | \vec{b} \rangle + \langle \vec{a} | t(\varepsilon) | - \vec{b} \rangle \tag{31}$$

We would like to mention that the so obtained F-Y amplitudes fulfill the below symmetry relations, as can be seen from Eq. (30):

$$\langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | \psi_1 \rangle \;=\; \langle -\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | \psi_1 \rangle$$

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \psi_2 \rangle \;=\; \langle -\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \psi_2 \rangle$$

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \psi_2 \rangle \;=\; \langle \vec{v}_1 \, \vec{v}_2 \, - \vec{v}_3 | \psi_2 \rangle \tag{32}$$

From the F-Y components $|\psi_1\rangle$ and $|\psi_2\rangle$ the four-body wave function is obtained by adding the components defined in different $3+1$ and $2+2$ type chains as given in Eq. (9). After evaluating the permutation operators P, $\tilde{P}$ and P_{34} the wave function is given as:

$$|\Psi\rangle = |\Psi_1\rangle + |\Psi_2\rangle \tag{33}$$

Where $|\Psi_1\rangle$ $(|\Psi_2\rangle)$ is corresponding to all $3+1$ $(2+2)$ type chains:

$$\langle \vec{u}_1 \ \vec{u}_2 \ \vec{u}_3|\Psi_1\rangle \ =$$

$$\{ \ \langle \vec{u}_1 \ \vec{u}_2 \ \vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{-1}{2}\vec{u}_1 - \frac{3}{4}\vec{u}_2 \ \vec{u}_1 - \frac{1}{2}\vec{u}_2 \ \vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{-1}{2}\vec{u}_1 + \frac{3}{4}\vec{u}_2 \ -\vec{u}_1 - \frac{1}{2}\vec{u}_2 \ \vec{u}_3|\psi_1\rangle \}^{123+4}$$

$$+\{ \ \langle \vec{u}_1 \ \frac{1}{3}\vec{u}_2 + \frac{8}{9}\vec{u}_3 \ \vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{-1}{2}\vec{u}_1 - \frac{1}{4}\vec{u}_2 - \frac{2}{3}\vec{u}_3 \ \vec{u}_1 - \frac{1}{6}\vec{u}_2 - \frac{4}{9}\vec{u}_3 \ \vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{-1}{2}\vec{u}_1 + \frac{1}{4}\vec{u}_2 + \frac{2}{3}\vec{u}_3 \ -\vec{u}_1 - \frac{1}{6}\vec{u}_2 - \frac{4}{9}\vec{u}_3 \ \vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle \}^{124+3}$$

$$+\{ \ \langle \frac{1}{2}\vec{u}_1 - \frac{3}{4}\vec{u}_2 \ -\frac{1}{3}\vec{u}_1 - \frac{1}{6}\vec{u}_2 + \frac{8}{9}\vec{u}_3 \ -\vec{u}_1 - \frac{1}{2}\vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{1}{2}\vec{u}_2 - \frac{2}{3}\vec{u}_3 \ \frac{2}{3}\vec{u}_1 - \frac{2}{3}\vec{u}_2 - \frac{4}{9}\vec{u}_3 \ -\vec{u}_1 - \frac{1}{2}\vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{-1}{2}\vec{u}_1 + \frac{1}{4}\vec{u}_2 + \frac{2}{3}\vec{u}_3 \ -\frac{1}{3}\vec{u}_1 + \frac{5}{6}\vec{u}_2 - \frac{4}{9}\vec{u}_3 \ -\vec{u}_1 - \frac{1}{2}\vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle \}^{134+2}$$

$$+\{ \ \langle -\frac{1}{2}\vec{u}_1 - \frac{3}{4}\vec{u}_2 \ \frac{1}{3}\vec{u}_1 - \frac{1}{6}\vec{u}_2 + \frac{8}{9}\vec{u}_3 \ \vec{u}_1 - \frac{1}{2}\vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{1}{2}\vec{u}_2 - \frac{2}{3}\vec{u}_3 \ -\frac{2}{3}\vec{u}_1 - \frac{2}{3}\vec{u}_2 - \frac{4}{9}\vec{u}_3 \ \vec{u}_1 - \frac{1}{2}\vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle$$

$$+\langle \frac{1}{2}\vec{u}_1 + \frac{1}{4}\vec{u}_2 + \frac{2}{3}\vec{u}_3 \ \frac{1}{3}\vec{u}_1 + \frac{5}{6}\vec{u}_2 - \frac{4}{9}\vec{u}_3 \ \vec{u}_1 - \frac{1}{2}\vec{u}_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle \}^{234+1} \tag{34}$$

$$\langle \vec{v}_1 \ \vec{v}_2 \ \vec{v}_3|\Psi_2\rangle \ =$$

$$\{ \ \langle \vec{v}_1 \ \vec{v}_2 \ \vec{v}_3|\psi_2\rangle$$

$$+\langle \vec{v}_3 \ -\vec{v}_2 \ \vec{v}_1|\psi_2\rangle \}^{12+34}$$

$$+\{ \ \langle \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 - \frac{1}{2}\vec{v}_3 \ \vec{v}_1 + \vec{v}_3 \ -\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3|\psi_2\rangle$$

$$+\langle \frac{-1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3 \ -\vec{v}_1 - \vec{v}_3 \ \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 - \frac{1}{2}\vec{v}_3|\psi_2\rangle \}^{13+24}$$

$$+\{ \ \langle \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3 \ \vec{v}_1 - \vec{v}_3 \ -\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 - \frac{1}{2}\vec{v}_3|\psi_2\rangle$$

$$+\langle -\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 - \frac{1}{2}\vec{v}_3 \ -\vec{v}_1 + \vec{v}_3 \ \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3|\psi_2\rangle \}^{14+23} \tag{35}$$

Each curly bracket contains all possible chains in the subsystem which is indicated with corresponding superscript. Already here we see that:

$$\langle \vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | \Psi_1 \rangle = \langle -\vec{u}_1 \, \vec{u}_2 \, \vec{u}_3 | \Psi_1 \rangle$$

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \Psi_2 \rangle = \langle -\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \Psi_2 \rangle$$

$$\langle \vec{v}_1 \, \vec{v}_2 \, \vec{v}_3 | \Psi_2 \rangle = \langle \vec{v}_1 \, \vec{v}_2 \, -\vec{v}_3 | \Psi_2 \rangle \tag{36}$$

Eq. (36) is satisfied if the F-Y components fulfill the expected symmetries in Eq. (32).

IV.   CHOOSING THE COORDINATE SYSTEMS

The F-Y components $|\psi_i(\vec{A}_1 \, \vec{A}_2 \, \vec{A}_3)\rangle$ are given as a function of Jacobi momenta vectors and as a solution of coupled three-dimensional integral equations, Eq. (30). Since we ignore spin and isospin dependencies, the both F-Y components $|\psi_i(\vec{A}_1 \, \vec{A}_2 \, \vec{A}_3)\rangle$ are scalars and thus only depend on the magnitudes of Jacobi momenta and the angles between them. The first important step for an explicit calculation is the selection of independent variables. One needs six variables to uniquely specify the geometry of the three vectors $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$, which are shown in Fig. 1. Having in mind that with three vectors one can span 2 planes, i.e. the $\vec{A}_3 - \vec{A}_1$ plane and $\vec{A}_3 - \vec{A}_2$ plane, a natural choice of independent variables is [53]:

$$A_1 = |\vec{A}_1| \quad A_2 = |\vec{A}_2| \quad A_3 = |\vec{A}_3|$$
$$x_1 = \hat{\vec{A}}_3 . \hat{\vec{A}}_1 \quad x_2 = \hat{\vec{A}}_3 . \hat{\vec{A}}_2 \quad x_{12}^3 = (\widehat{\vec{A}_3 \times \vec{A}_1}) . (\widehat{\vec{A}_3 \times \vec{A}_2}) \tag{37}$$

The last variable, x_{12}^3, is the angle between the two normal vectors of the $\vec{A}_3 - \vec{A}_1$ plane and the $\vec{A}_3 - \vec{A}_2$ plane, which is explicitly related to the angle between $\vec{A}_1$ and $\vec{A}_2$ vectors as:

$$x_{12}^3 = \frac{\hat{\vec{A}}_1 . \hat{\vec{A}}_2 - (\hat{\vec{A}}_3 . \hat{\vec{A}}_1)(\hat{\vec{A}}_3 . \hat{\vec{A}}_2)}{\sqrt{1 - (\hat{\vec{A}}_3 . \hat{\vec{A}}_1)^2} \sqrt{1 - (\hat{\vec{A}}_3 . \hat{\vec{A}}_2)^2}} \tag{38}$$

Therefore in order to solve Eq. (30) directly without employing PW projection, we have to define suitable coordinate systems. As shown in Fig. 2, for both F-Y components we choose the third vector parallel to $Z-$axis, the second vector in the $X - Z$ plane and express the remaining vectors, the first as well as the integration vectors, with respect to them. We have the magnitudes of vectors as well as the following angle relations as variables:
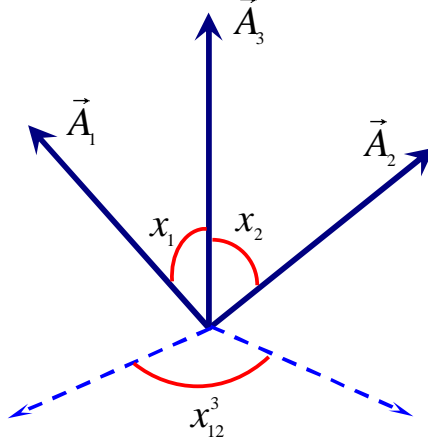
FIG. 1: The geometry of three vectors $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$ relevant in the four-body bound state problem. The independent angle variables x_1, x_2 and x_{12}^3 as defined in Eq. (37) are indicated. The dashed arrows represent the normal vectors $(\vec{A}_3 \times \vec{A}_1)$ and $(\vec{A}_3 \times \vec{A}_2)$.
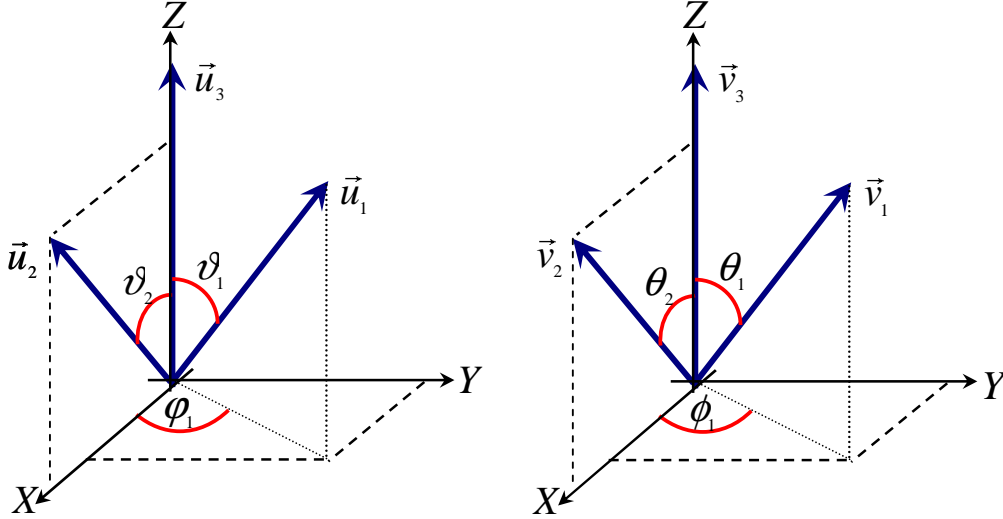

FIG. 2: These figures show the geometry of both vector sets $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ relevant in the F-Y componets. The third vectors $\vec{u}_3$, $\vec{v}_3$ have been chosen parallel to $Z-$axis, the second vectors $\vec{u}_2$, $\vec{v}_2$ in the $X - Z$ plane and the first vectors $\vec{u}_1$, $\vec{v}_1$ are free in the space. The independent angle variables x_1, x_2, x_{12}^3 and X_1, X_2, X_{12}^3 as defined in Eqs. (39) and (40) are indicated.

$$
\begin{aligned}
x_1 &= \hat{\tilde{u}}_3.\hat{\tilde{u}}_1 \equiv \cos(\vartheta_1) \\
x_2 &= \hat{\tilde{u}}_3.\hat{\tilde{u}}_2 \equiv \cos(\vartheta_2) \\
x_{12}^3 &= \hat{\tilde{u}}_1^{xy}.\hat{\tilde{u}}_2^{xy} \equiv \cos(\varphi_1) \\
y_{12} &= \hat{\tilde{u}}_1.\hat{\tilde{u}}_2 \equiv x_1 x_2 + \sqrt{1-x_1^2}\sqrt{1-x_2^2}\cos(\varphi_1) \\
x_2' &= \hat{\tilde{u}}_3.\hat{\tilde{u}}_2' \equiv \cos(\vartheta_2') \\
y_{12'} &= \hat{\tilde{u}}_1.\hat{\tilde{u}}_2' \equiv x_1 x_2' + \sqrt{1-x_1^2}\sqrt{1-x_2'^2}\cos(\varphi_1-\varphi_2') \\
y_{22'} &= \hat{\tilde{u}}_2.\hat{\tilde{u}}_2' \equiv x_2 x_2' + \sqrt{1-x_2^2}\sqrt{1-x_2'^2}\cos(\varphi_2')
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
X_1 &= \hat{\tilde{v}}_3.\hat{\tilde{v}}_1 \equiv \cos(\theta_1) \\
X_2 &= \hat{\tilde{v}}_3.\hat{\tilde{v}}_2 \equiv \cos(\theta_2) \\
X_{12}^3 &= \hat{\tilde{v}}_1^{xy}.\hat{\tilde{v}}_2^{xy} \equiv \cos(\phi_1) \\
Y_{12} &= \hat{\tilde{v}}_1.\hat{\tilde{v}}_2 \equiv X_1 X_2 + \sqrt{1-X_1^2}\sqrt{1-X_2^2}\cos(\phi_1) \\
X_3' &= \hat{\tilde{v}}_3.\hat{\tilde{v}}_3' \equiv \cos(\theta_3') \\
Y_{13'} &= \hat{\tilde{v}}_1.\hat{\tilde{v}}_3' \equiv X_1 X_3' + \sqrt{1-X_1^2}\sqrt{1-X_3'^2}\cos(\phi_1-\phi_3') \\
Y_{23'} &= \hat{\tilde{v}}_2.\hat{\tilde{v}}_3' \equiv X_2 X_3' + \sqrt{1-X_2^2}\sqrt{1-X_3'^2}\cos(\phi_3')
\end{aligned}
\tag{40}
$$

With this choice of variables the matrix elements of F-Y components are given as:

$$
\begin{aligned}
\langle \vec{u}_1\, \vec{u}_2\, \vec{u}_3 | \psi_1 \rangle &\equiv \psi_1(u_1\, u_2\, u_3\, x_1\, x_2\, x_{12}^3) \\
\langle \vec{v}_1\, \vec{v}_2\, \vec{v}_3 | \psi_2 \rangle &\equiv \psi_2(v_1\, v_2\, v_3\, X_1\, X_2\, X_{12}^3)
\end{aligned}
\tag{41}
$$

Furthermore $\langle \vec{u}_1 | t_s(\epsilon) | \frac{1}{2}\vec{u}_2 + \vec{u}_2' \rangle$ and $\langle \vec{v}_1 | t_s(\epsilon^*) | \vec{v}_3' \rangle$ are also scalar functions, and then can be written in the following form:

$$
\begin{aligned}
\langle \vec{u}_1 | t_s(\epsilon) | \frac{1}{2}\vec{u}_2 + \vec{u}_2' \rangle &\equiv t_s(\vec{u}_1, \frac{1}{2}\vec{u}_2 + \vec{u}_2'; \epsilon) \equiv t_s(u_1, \tilde{\pi}, \tilde{x}; \epsilon) \\
\langle \vec{v}_1 | t_s(\epsilon^*) | \vec{v}_3' \rangle &\equiv t_s(\vec{v}_1, \vec{v}_3'; \epsilon^*) \equiv t_s(v_1, v_3', Y_{13'}; \epsilon^*)
\end{aligned}
\tag{42}
$$

Where

$$
\begin{aligned}
\tilde{\pi} &= |\frac{1}{2}\vec{u}_2 + \vec{u}_2'| = \sqrt{\frac{1}{4}u_2^2 + u_2'^2 + u_2 u_2' y_{22'}} \\
\tilde{x} &= \hat{\tilde{u}}_1.(\widehat{\frac{1}{2}\vec{u}_2 + \vec{u}_2'}) = \frac{1}{\tilde{\pi}}(\frac{1}{2}u_2 y_{12} + u_2' y_{12'})
\end{aligned}
\tag{43}
$$

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The more complex dependencies appear under the integrals in Eq. (30) for magnitude and angle variables of F-Y components. According to Eq. (37) and Eq. (38) they are given as:

$$\langle \vec{u}_2 + \frac{1}{2}\vec{u}\,'_2 \ \vec{u}\,'_2 \ \vec{u}_3|\psi_1\rangle \equiv \psi_1(\pi_1 \ u'_2 \ u_3 \ x_{12} \ x_{13} \ x^{u_3}_{\pi_1 u'_2})$$

$$\langle \vec{u}_2 + \frac{1}{2}\vec{u}\,'_2 \ \frac{1}{3}\vec{u}\,'_2 + \frac{8}{9}\vec{u}_3 \ \vec{u}\,'_2 - \frac{1}{3}\vec{u}_3|\psi_1\rangle \equiv \psi_1(\pi_1 \ \pi_2 \ \pi_3 \ x_{22} \ x_{23} \ x^{\pi_3}_{\pi_1 \pi_2})$$

$$\langle \vec{u}_2 + \frac{1}{2}\vec{u}\,'_2 \ - \vec{u}\,'_2 - \frac{2}{3}\vec{u}_3 \ \frac{1}{2}\vec{u}\,'_2 - \frac{2}{3}\vec{u}_3|\psi_2\rangle \equiv \psi_2(\pi_1 \ \pi_4 \ \pi_5 \ x_{32} \ x_{33} \ x^{\pi_5}_{\pi_1 \pi_4})$$

$$\langle \vec{v}_3 \ \frac{2}{3}\vec{v}_2 + \frac{2}{3}\vec{v}\,'_3 \ \frac{1}{2}\vec{v}_2 - \vec{v}\,'_3|\psi_1\rangle \equiv \psi_1(v_3 \ \Sigma_1 \ \Sigma_2 \ X_{12} \ X_{13} \ X^{\Sigma_2}_{v_3\Sigma_1})$$

$$\langle \vec{v}_3 \ - \vec{v}_2 \ \vec{v}\,'_3|\psi_2\rangle \equiv \psi_2(v_3 \ v_2 \ v'_3 \ X_{22} \ X_{23} \ X^{v'_3}_{v_3 v_2}) \tag{44}$$

Where the shifted arguments are:

$$\pi_1 = |\vec{u}_2 + \frac{1}{2}\vec{u}\,'_2| = \sqrt{u_2^2 + \frac{1}{4}u_2'^2 + u_2 u'_2 y_{22'}}$$

$$\pi_2 = |\frac{1}{3}\vec{u}\,'_2 + \frac{8}{9}\vec{u}_3| = \sqrt{\frac{1}{9}u_2'^2 + \frac{64}{81}u_3^2 + \frac{16}{27}u'_2 u_3 x'_2}$$

$$\pi_3 = |\vec{u}\,'_2 - \frac{1}{3}\vec{u}_3| = \sqrt{u_2'^2 + \frac{1}{9}u_3^2 - \frac{2}{3}u'_2 u_3 x'_2}$$

$$\pi_4 = |-\vec{u}\,'_2 - \frac{2}{3}\vec{u}_3| = \sqrt{u_2'^2 + \frac{4}{9}u_3^2 + \frac{4}{3}u'_2 u_3 x'_2}$$

$$\pi_5 = |\frac{1}{2}\vec{u}\,'_2 - \frac{2}{3}\vec{u}_3| = \sqrt{\frac{1}{4}u_2'^2 + \frac{4}{9}u_3^2 - \frac{2}{3}u'_2 u_3 x'_2}$$

$$\Sigma_1 = |\frac{2}{3}\vec{v}_2 + \frac{2}{3}\vec{v}\,'_3| = \frac{2}{3}\sqrt{v_2^2 + v_3'^2 + 2v_2 v'_3 Y_{23'}}$$

$$\Sigma_2 = |\frac{1}{2}\vec{v}_2 - \vec{v}\,'_3| = \sqrt{\frac{1}{4}v_2^2 + v_3'^2 - v_2 v'_3 Y_{23'}} \tag{45}$$

$$x_{11} = (\widehat{\vec{u}_2 + \frac{1}{2}\vec{u}\,'_2}).\hat{\vec{u}}\,'_2 = \frac{1}{\pi_1}(u_2 y_{22'} + \frac{1}{2}u'_2)$$

$$x_{12} = (\widehat{\vec{u}_2 + \frac{1}{2}\vec{u}\,'_2}).\hat{\vec{u}}_3 = \frac{1}{\pi_1}(u_2 x_2 + \frac{1}{2}u'_2 x'_2)$$

$$x_{13} = \hat{\vec{u}}\,'_2.\hat{\vec{u}}_3 = x'_2$$

$$x^{u_3}_{\pi_1 u'_2} = \frac{x_{11} - x_{12}x_{13}}{\sqrt{1 - x_{12}^2}\sqrt{1 - x_{13}^2}} \tag{46}$$

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$$x_{21} = \widehat{(\vec{u}_2 + \frac{1}{2}\vec{u}'_2)}.\widehat{(\frac{1}{3}\vec{u}'_2 + \frac{8}{9}\vec{u}_3)}$$

$$= \frac{1}{\pi_1\pi_2}(\frac{1}{3}u_2u'_2y_{22'} + \frac{8}{9}u_2u_3x_2 + \frac{1}{6}u'^2_2 + \frac{4}{9}u'_2u_3x'_2)$$

$$x_{22} = \widehat{(\vec{u}_2 + \frac{1}{2}\vec{u}'_2)}.\widehat{(\vec{u}'_2 - \frac{1}{3}\vec{u}_3)}$$

$$= \frac{1}{\pi_1\pi_3}(u_2u'_2y_{22'} - \frac{1}{3}u_2u_3x_2 + \frac{1}{2}u'^2_2 - \frac{1}{6}u'_2u_3x'_2)$$

$$x_{23} = \widehat{(\frac{1}{3}\vec{u}'_2 + \frac{8}{9}\vec{u}_3)}.\widehat{(\vec{u}'_2 - \frac{1}{3}\vec{u}_3)}$$

$$= \frac{1}{\pi_2\pi_3}(\frac{1}{3}u'_2 + \frac{7}{9}u'_2u_3x'_2 - \frac{8}{27}u^2_3)$$

$$x^{\pi_3}_{\pi_1\pi_2} = \frac{x_{21} - x_{22}x_{23}}{\sqrt{1 - x^2_{22}}\sqrt{1 - x^2_{23}}} \tag{47}$$

$$x_{31} = \widehat{(\vec{u}_2 + \frac{1}{2}\vec{u}'_2)}.\widehat{(-\vec{u}'_2 - \frac{2}{3}\vec{u}_3)}$$

$$= \frac{-1}{\pi_1\pi_4}(u_2u'_2y_{22'} + \frac{2}{3}u_2u_3x_2 + \frac{1}{2}u'^2_2 + \frac{1}{3}u'_2u_3x'_2)$$

$$x_{32} = \widehat{(\vec{u}_2 + \frac{1}{2}\vec{u}'_2)}.\widehat{(\frac{1}{2}\vec{u}'_2 - \frac{2}{3}\vec{u}_3)}$$

$$= \frac{1}{\pi_1\pi_5}(\frac{1}{2}u_2u'_2y_{22'} - \frac{2}{3}u_2u_3x_2 + \frac{1}{4}u'^2_2 - \frac{1}{3}u'_2u_3x'_2)$$

$$x_{33} = \widehat{(-\vec{u}'_2 - \frac{2}{3}\vec{u}_3)}.\widehat{(\frac{1}{2}\vec{u}'_2 - \frac{2}{3}\vec{u}_3)}$$

$$= \frac{-1}{\pi_4\pi_5}(\frac{1}{2}u'_2 - \frac{1}{3}u'_2u_3x'_2 - \frac{4}{9}u^2_3)$$

$$x^{\pi_5}_{\pi_1\pi_4} = \frac{x_{31} - x_{32}x_{33}}{\sqrt{1 - x^2_{32}}\sqrt{1 - x^2_{33}}} \tag{48}$$

$$X_{11} = \hat{\vec{v}}_3.\widehat{(\frac{2}{3}\vec{v}_2 + \frac{2}{3}\vec{v}'_3)} = \frac{\frac{2}{3}}{\Sigma_1}(v_2X_2 + v'_3X'_3)$$

$$X_{12} = \hat{\vec{v}}_3.\widehat{(\frac{1}{2}\vec{v}_2 - \vec{v}'_3)} = \frac{1}{\Sigma_2}(\frac{1}{2}v_2X_2 - v'_3X'_3)$$

$$X_{13} = \widehat{(\frac{2}{3}\vec{v}_2 + \frac{2}{3}\vec{v}'_3)}.\widehat{(\frac{1}{2}\vec{v}_2 - \vec{v}'_3)} = \frac{\frac{2}{3}}{\Sigma_1\Sigma_2}(\frac{1}{2}v^2_2 - \frac{1}{2}v_2v'_3Y_{23'} - v'^2_3)$$

$$X^{\Sigma_2}_{v_3\Sigma_1} = \frac{X_{11} - X_{12}X_{13}}{\sqrt{1 - X^2_{12}}\sqrt{1 - X^2_{13}}} \tag{49}$$

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$$X_{21} = \hat{\vec{v}}_3.(-\hat{\vec{v}}_2) = -X_2$$

$$X_{22} = \hat{\vec{v}}_3.\hat{\vec{v}}_3' = X_3'$$

$$X_{23} = (-\hat{\vec{v}}_2).\hat{\vec{v}}_3' = -Y_{23'}$$

$$X_{v_3 v_2}^{v_3'} = \frac{X_{21} - X_{22}X_{23}}{\sqrt{1 - X_{22}^2}\sqrt{1 - X_{23}^2}} \tag{50}$$

These considerations lead to the explicit representation for the F-Y components $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$
\begin{aligned}
\psi_1(u_1\, u_2\, u_3\, x_1\, x_2\, x_{12}^3) =\ & \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \int_0^\infty du_2'\, u_2'^2 \int_{-1}^{+1} dx_2' \int_0^{2\pi} d\varphi_2' \\
& t_s(u_1, \tilde{\pi}, \tilde{x}; \epsilon) \\
& \times\ \{\ \psi_1(\pi_1\ u_2'\ u_3\ x_{12}\ x_{13}\ x_{\pi_1 u_2'}^{u_3}) \\
& \quad + \psi_1(\pi_1\ \pi_2\ \pi_3\ x_{22}\ x_{23}\ x_{\pi_1 \pi_2}^{\pi_3}) \\
& \quad + \psi_2(\pi_1\ \pi_4\ \pi_5\ x_{32}\ x_{33}\ x_{\pi_1 \pi_4}^{\pi_5})\ \}
\end{aligned}
$$

$$
\begin{aligned}
\psi_2(v_1\, v_2\, v_3\, X_1\, X_2\, X_{12}^3) =\ & \frac{\frac{1}{2}}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \int_0^\infty dv_3'\, v_3'^2 \int_{-1}^{+1} dX_3' \int_0^{2\pi} d\phi_3' \\
& t_s(v_1, v_3', Y_{13'}; \epsilon^*) \\
& \times\ \{\ 2\,\psi_1(v_3\ \Sigma_1\ \Sigma_2\ X_{12}\ X_{13}\ X_{v_3 \Sigma_1}^{\Sigma_2}) \\
& \quad + \psi_2(v_3\ v_2\ v_3'\ X_{22}\ X_{23}\ X_{v_3 v_2}^{v_3'})\ \}
\end{aligned} \tag{51}
$$

The above coupled equations, Eq. (51), is the starting point for numerical calculations, and the details will be described in the next section. In a standard PW representation Eq. (15) is replaced by two coupled sets of finite number of coupled integral equations [40]:

$$
\begin{aligned}
\langle u_1\, u_2\, u_3\, \alpha_1|\psi_1\rangle \;=\; & \sum_{\alpha_1'}\int D^3 u' \sum_{\alpha_1''}\int D^3 u'' \sum_{\alpha_1'''}\int D^3 u''' \\
& \langle u_1\, u_2\, u_3\, \alpha_1|G_0 t|u_1'\, u_2'\, u_3'\, \alpha_1'\rangle \, \langle u_1'\, u_2'\, u_3'\, \alpha_1'|P|u_1''\, u_2''\, u_3''\, \alpha_1''\rangle \\
& \langle u_1''\, u_2''\, u_3''\, \alpha_1''|(1+P_{34})|u_1'''\, u_2'''\, u_3'''\, \alpha_1'''\rangle \langle u_1'''\, u_2'''\, u_3'''\, \alpha_1'''|\psi_1\rangle \\
+\; & \sum_{\alpha_1'}\int D^3 u' \sum_{\alpha_1''}\int D^3 u'' \sum_{\alpha_2'}\int D^3 v' \\
& \langle u_1\, u_2\, u_3\, \alpha_1|G_0 t|u_1'\, u_2'\, u_3'\, \alpha_1'\rangle \, \langle u_1'\, u_2'\, u_3'\, \alpha_1'|P|u_1''\, u_2''\, u_3''\, \alpha_1''\rangle \\
& \langle u_1''\, u_2''\, u_3''\, \alpha_1''|v_1'\, v_2'\, v_3'\, \alpha_2'\rangle \, \langle v_1'\, v_2'\, v_3'\, \alpha_2'|\psi_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle v_1\, v_2\, v_3\, \alpha_2|\psi_2\rangle \;=\; & \sum_{\alpha_2'}\int D^3 v' \sum_{\alpha_2''}\int D^3 v'' \sum_{\alpha_1'}\int D^3 u' \\
& \langle v_1\, v_2\, v_3\, \alpha_2|G_0 t|v_1'\, v_2'\, v_3'\, \alpha_2'\rangle \, \langle v_1'\, v_2'\, v_3'\, \alpha_2'|\tilde{P}|v_1''\, v_2''\, v_3''\, \alpha_2''\rangle \\
& \langle v_1''\, v_2''\, v_3''\, \alpha_2''|u_1'\, u_2'\, u_3'\, \alpha_1'\rangle \langle u_1'\, u_2'\, u_3'\, \alpha_1'|(1+P_{34})\psi_1\rangle \\
+\; & \sum_{\alpha_2'}\int D^3 v' \sum_{\alpha_2''}\int D^3 v'' \, \langle v_1\, v_2\, v_3\, \alpha_2|G_0 t|v_1'\, v_2'\, v_3'\, \alpha_2'\rangle \\
& \langle v_1'\, v_2'\, v_3'\, \alpha_2'|\tilde{P}|v_1''\, v_2''\, v_3''\, \alpha_2''\rangle \langle v_1''\, v_2''\, v_3''\, \alpha_2''|\psi_2\rangle
\end{aligned} \tag{52}
$$

Where $\alpha_1 \equiv (l_1 l_2)j_3, (j_3 l_3); J = 0$ and $\alpha_2 \equiv (\lambda_1 \lambda_2)I, (I\lambda_3); J = 0$. Here the evaluation of two-body $t-$matrices and permutation operators $P, \tilde{P}$ and P_{34} as well as coordinate transformations due to considering angular momentum quantum numbers instead of angle variables lead to more complicated expressions in comparison to our 3D representation.

V. NUMERICAL TECHNIQUES

In this section we describe the details of the numerical algorithm for solving the coupled F-Y three-dimensional integral equations, Eq. (51). The coupled F-Y equations Eq. (51), represent a set of three-dimensional homogenous integral equations, which after discreatization turns into a huge matrix eigenvalue equation. The dependence on the continuous momentum and angle variables $(u_i, v_i; i = 1, 2, 3$ and $x_1, x_2, x_{12}^3, X_1, X_2, X_{12}^3)$ is replaced in the numerical treatment by a dependence on certain discrete values. Let the numbers of these discrete points be denoted by N_{jac}, N_{sph} and N_{pol} corresponding to momentum $(u_i, v_i; i = 1, 2, 3)$, spherical angle (x_1, x_2, X_1, X_2) and polar angle (x_{12}^3, X_{12}^3) variables, the

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dimension of the eigenvalue problem is:

$$N = N_{jac}^3 \times N_{sph}^2 \times N_{pol} \times 2 \tag{53}$$

The huge matrix eigenvalue equation requires an iterative solution method. We use a Lanczos-like scheme, the method of iterated orthonormal vectors (IOV) that is proved to be very efficient for nuclear few-body problems [58]. This technique reduces the dimension of the eigenvalue problem to the number of iteration minus one. The eigenvalue equation, Eq. (51), schematically can be written as:

$$\lambda(E)\,\psi = K(E)\,\psi \tag{54}$$

The kernel of the linear equations $K(E)$ is energy dependent, and $\lambda(E)$ is its eigenvalue with ψ as the corresponding eigenvector. ψ represents the set of F-Y components as $\psi = \binom{\psi_1}{\psi_2}$. For the physical binding energy the eigenvalue $\lambda(E)$ of the matrix kernel $K(E)$ has to be one. We start the iteration with two gaussian F-Y components and stop the iteration after 5-10 times. In order to solve the eigenvalue equation, Eq. (54), for the F-Y components, Eq. (51), we use the Gaussian quadrature grid points for the momentum and angle variables.

The momentum variables have to cover the interval $[0, \infty]$. Because the F-Y components drop sufficiently rapidly we limit the intervals to suitable cut-offs. We choose the cut-offs as $u_1^{max} = v_1^{max} = v_3^{max} = v_3'^{max}$, $u_2^{max} = u_2'^{max}$ and $u_3^{max} = v_2^{max}$. These cut-off values vary depending on the potential we use but they are chosen large enough to achieve cut-off independence.

The iteration of Eq. (51) requires a three-dimensional and a six-dimensional interpolation on $\psi_1(u_1\,u_2\,u_3\,x_1\,x_2\,x_{12}^3)$ and a six dimensional interpolation on $\psi_2(v_1\,v_2\,v_3\,X_1\,X_2\,X_{12}^3)$ for the first F-Y component. Also it requires a five-dimensional interpolation on $\psi_1(u_1\,u_2\,u_3\,x_1\,x_2\,x_{12}^3)$ and a two-dimensional interpolation on $\psi_2(v_1\,v_2\,v_3\,X_1\,X_2\,X_{12}^3)$ for the second F-Y component. The interpolations should be carry out in the shifted momentum and angle arguments. By adding the additional grid points, 0 to all momentum and ± 1 to all angle grid points, we avoid the extrapolation outside the Gaussian grids.

We would like to point out that the symmetry properties as shown in Eq. (32) can be implemented in the iteration of Eq. (51) to cut down the size of the F-Y components and thus save time and memory when computing the integrals.

The functional behavior of $K(E)$ is determined by the two-body $t-$matrices $t_s(u_1, \tilde{\pi}, \tilde{x}; \epsilon)$

and $t_s(v_1, v'_3, Y_{13'}; \epsilon^*)$. We solve the Lippmann-Schwinger equation for the fully-off-shell two-body t−matrices directly as function of the Jacobi vector variables as described in ref. [42]. The so obtained t−matrices are then symmetrized to get $t_s(u_1, \widetilde{u}_1, \widetilde{x}; \epsilon)$ and $t_s(v_1, v'_3, \widetilde{X}; \epsilon^*)$, where $\tilde{\pi}, \tilde{x}$ and $Y_{13'}$ in Eq. (51) are replaced with new momentum and angle mesh grids $\widetilde{u}_1, \widetilde{x}$ and $\widetilde{X}$. Both angle mesh grids $\widetilde{x}$ and $\widetilde{X}$ cover interval $[-1, +1]$ and momentum mesh grid $\widetilde{u}_1$ covers $[0, \tilde{\pi}^{max}]$. We would like to point out that after having t−matrix $t_s(u_1, \widetilde{u}_1, \widetilde{x}; \epsilon)$ $(t_s(v_1, v'_3, \widetilde{X}; \epsilon^*))$ on grids for $u_1, \widetilde{u}_1$ and $\widetilde{x}$ $(v_1, v'_3$ and $\widetilde{X})$ we solve the integral equation again to obtain t−matrix at extra points $u_1 = 0$ and $\widetilde{x} = \pm 1$ $(v_1 = 0$ and $\widetilde{X} = \pm 1)$. Thus when iterating Eq. (51) we do not have to extrapolate numerically to first momentum $u_1(v_1)$ and angle variable $\widetilde{x}(\widetilde{X})$ of $t_s(u_1, \widetilde{u}_1, \widetilde{x}; \epsilon)$ $(t_s(v_1, v'_3, \widetilde{X}; \epsilon^*))$. Also we point out that the momentum dependencies given in Eq. (51) suggest that we calculate the two-body t−matrix $t_s(u_1, \widetilde{u}_1, \widetilde{x}; \epsilon)$ $(t_s(v_1, v'_3, \widetilde{X}; \epsilon^*))$ for the energies $\epsilon = E - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}$ $(\epsilon^* = E - \frac{v_2^2}{2m} - \frac{v_3^2}{m})$ dictated by the same u_2 and u_3 grids (v_2 and v_3 grids). So each energy depends on two momentum variables. The number of different energies, where both t−matrices are needed, is quadratic in the number of momentum grid points. Consequently both t−matrices would be extremely huge if we keep the dependence on both momenta. Therefore we introduce two additional energy grids $\hat{\epsilon}$ and $\hat{\epsilon}^*$ and insert an interpolation step from these grids to ϵ and ϵ^*. This reduce the memory and computing time necessary for both t−matrices tremendously. In order to obtain the second momentum, the angle and the energy for $t_s(u_1, \widetilde{u}_1, \widetilde{x}; \hat{\epsilon})$, also the angle and the energy for $t_s(v_1, v'_3, \widetilde{X}; \hat{\epsilon}^*)$ required in the iteration of Eq. (51), we have to carry out three- and two-dimensional interpolations respectfully.

Since the coupled integral equations, Eq. (51), require a very large number of interpolations, we use the cubic Hermitian splines of ref. [59] for its accuracy and high computational speed. It can be useful to mention that in the numerical calculations we use the Lapack library [60], for solving a system of linear equations in calculation of the two-body t−matrices, and Arpack library [61], for solving the eigenvalue problem.

VI. NUMERICAL RESULTS

A. Three- and Four-Body Binding Energies

In order to be able to compare our calculations with results obtained by other techniques we use the following spin-independent potentials:

Gauss-type Baker potential [62]

$$V(r) = -51.5 \, e^{-0.3906 \, r^2} \quad [MeV] \tag{55}$$

Gauss-type Volkov potential [63]

$$V(r) = 144.86 \, e^{-1.487 \, r^2} - 83.34 \, e^{-0.3906 \, r^2} \quad [MeV] \tag{56}$$

separable Yamaguchi potential [64]

$$V(p, p') = -\frac{\lambda}{m} \, g(p) \, g(p') \, ; \; g(p) = \frac{1}{p^2 + \beta^2} \tag{57}$$

and the spin-averaged Yukawa-type Malfliet-Tjon V potential [65]

$$V(r) = 1458.05 \frac{e^{-3.11r}}{r} - 578.09 \frac{e^{-1.55r}}{r} \quad [MeV] \tag{58}$$

The parameters used for Yamaguchi potentials are given in table 1. In our calculations with above potentials we use $m^{-1} = 41.470 \, MeV.fm^2$. For four-body(three-body) binding energy calculations twenty(thirty two) grid points for angle variables and thirty(forty) grid points for Jacobi momentum variables have been used respectively.

TABLE I: Parameters of the Yamaguchi-type potentials.

potential	$\lambda[fm^{-3}]$	$\beta[fm^{-1}]$
Y-I	0.415	1.45
Y-II	0.353	1.45
Y-III	0.182	1.15
Y-IV	0.179	1.15

The techniques to which we compare are the VAR [66]-[68] and HEE [69] methods, several types of approximating subsystem kernels of the four-body problem by operators of finite rank (SKFR) [70]-[72], the integrodifferential equation approach SIDE and IDEA [73], the

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CCE [74], the ATMS [75], the GFMC [76], the DFY [32],[77], the CRCGBV [78], the DMC [79] and last but not least 2DI [64],[80].

In table 2 we show the three- and four-body binding energies for Baker potential calculated with different methods. Our results for three- and four-body binding energies with values -9.76 and $-40.0 \, [MeV]$ are in good agreement with results of other available calculations.

TABLE II: Three- and Four-Body binding energies for Baker potential.

Method	4-body B.E. [MeV]	3-body B.E. [MeV]
VAR [66]	-39.1±0.1	
VAR [67]	-40.03	
HHE [69]	-40.05	
DFY [77]	-40.0	
DFY [32]	-39.9989	
FY(PW) [35]	-40.03	-9.76
FY(3D)	-40.0	-9.76

For Volkov potential our calculations for three- and four-body binding energies yield the values -8.43 and $-30.2 \, [MeV]$ which as shown in table 3 are also in good agreement with other calculations.

TABLE III: Three- and Four-Body binding energies for Volkov potential.

Method	4-body B.E. [MeV]	3-body B.E. [MeV]
HH [18]	-30.420	
SVM [9]	-30.424	
VAR [67]	-30.317	
HHE [69]	-30.3988	
DFY [77]	-30.2	
DFY [32]	-30.2467	
FY(PW) [35]	-30.27	-8.43
FY(3D)	-30.2	-8.43

The three- and four-body binding energies for separable Yamaguchi type potentials calculated with different methods are listed in table 4. Our results for three-body binding energies

for Yamaguchi I, II, III and IV are $-25.41, -12.45, -9.25, -8.53\,[MeV]$ and for four-body binding energies are $-89.8, -54.5, -38.2, -36.2\,[MeV]$, which are in good agreement with results of other methods, specially with 2DI.

TABLE IV: Four-Body binding energies for Yamaguchi type potentials. The numbers in parenthesis are three-body binding energies.

Method	Y-I	Y-II	Y-III	Y-IV
SKFR [70]	-84.66			
SKFR [71]	-90.10			
SKFR [72]	-89.74			
FY(PW) [35]	-89.90 (-25.41)			
2DI [64]	-89.6 (-25.40)	-54.5 (-12.45)	-38.3 (-9.24)	-36.3 (-8.51)
FY(3D)	-89.8 (-25.41)	-54.5 (-12.45)	-38.2 (-9.25)	-36.2 (-8.53)

TABLE V: Convergence of the four-body binding energy with increasing number of partial waves for Malfliet-Tjon V potential [40].

$l_1, \lambda_1, \lambda_3$	l_2	l_3	λ_2	$E_{ground}[MeV]$
0	0	0	0	-31.07
2	2	0	0	-31.11
4	4	0	0	-31.22
6	6	0	0	-31.23
4	6	2	0	-31.28
4	6	4	0	-31.31
4	6	6	0	-31.31
4	6	4	2	-31.34
4	6	4	4	-31.35
4	6	4	6	-31.35
6	6	4	4	-31.36
8	6	6	6	-31.36

As demonstrated in table 5, the calculation of the four-body binding energy using the Malfliet-Tjon V potential in PW scheme converges to value of $E = -31.36\,[MeV]$. Here

convergence is reached for $l_1, \lambda_1, \lambda_3 = 8$ and $l_2, l_3, \lambda_2 = 6$ [40], while the three-body binding energy for this potential converges to $-7.73\,[MeV]$ [35]. As shown in table 6 our calculations for Malfliet-Tjon V yield the value $-31.3\,[MeV]$ for four-body binding energy, which is in good agreement with recent HH [18], EIHH [29], F-Y(PW) [40] and SVM [9] results and with other calculations. Also our result for three-body binding energy with value $-7.74\,[MeV]$ is in good agreement with the obtained value $-7.73\,[MeV]$ of Faddeev calculations in PW scheme.

TABLE VI: Four-Body binding energies for Malfliet-Tjon V. The numbers in parenthesis are three-body binding energies.

Method	4-body B.E. [MeV]
CRCGBV [78]	-31.357
ATMS [75]	-31.36
GFMC [76]	-31.3±0.2
CCE [74]	-31.24
VAR [68]	-31.19±0.05
IDEA [73]	-30.98
DMC [79]	-31.5
HH [18]	-31.347
SVM [9]	-31.360
EIHH [29]	-31.358
FY(PW) [35],[40]	-31.36 (-7.73)
FY(3D)	-31.3 (-7.74)

As we can see from these comparisons to other calculations of the four-body binding energy based on PW decomposition, our results provide the same accuracy while the numerical procedure are actually easier to implement. In the 3D case there is only two coupled three-dimensional integral equations to be solved, whereas in the PW case one has two coupled sets of finite number of coupled equations with kernels containing relatively complicated geometrical expressions.

B. Test of Calculations

In this section we investigate the numerical stability of our algorithm and our 3D representation of Yakubovsky components. We specially investigate the stability of the eigenvalue of the Yakubovsky kernel with respect to the number of grid points for Jacobi momenta, polar and azimuthal angle variables. We also investigate the quality of our representation of the Yakubovsky components and consequently wave function by calculating the expectation value of the Hamiltonian operator. For these investigations we use the Malfliet-Tjon V potential.

In table 7 we present the obtained eigenvalue results for binding energy $E = -31.3\ MeV$ for different grids. We choose the number of grid points for Jacobi momenta as $N_{u_1} = N_{v_1} = N_{v_3} = N_{v'_3} = N^1_{jac}$ and $N_{u_2} = N_{u_3} = N_{v_2} = N^2_{jac}$. As demonstrated in this table, the calculation of the eigenvalue λ convergence to the value one for $N^1_{jac} = N^2_{jac} = 30$ and $N_{sph} = N_{pol} = 20$. It should be clear that the solution of coupled Yakubovsky three-dimensional integral equations, with six independent variables for the amplitudes, is much more time-consuming with respect to the solution of three-dimensional Faddeev integral equation [45], with three variables for the amplitude.

TABLE VII: Stability of the eigenvalue λ of Yakubovsky kernel with respect to the number of grid points in Jacobi momenta N^1_{jac}, N^2_{jac}, spherical angles N_{sph} and polar angles N_{pol}, where $E = -31.3 MeV$.

N^1_{jac}	N^2_{jac}	$N_{sph} = N_{pol}$	λ
20	20	20	0.987
26	20	20	0.992
26	26	20	0.995
30	26	20	0.998
30	30	20	1.000

The solution of coupled Yakubovsky three-dimensional integral equations in momentum space allows to estimate numerical errors reliably. With the binding energy E and the Yakubovsky components $|\psi_1\rangle$ and $|\psi_2\rangle$ available, we are able to calculate the total wave function $|\Psi\rangle$ from Eqs. (33)-(35) by considering the choice of coordinate systems which are represented by Eqs. (39)-(40). So in order to demonstrate the reliability of our calculations

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we can evaluate the expectation value of the Hamiltonian operator H and compare this value to the previously calculated binding energy of the eigenvalue equation, Eq. (54). Explicitly we evaluate the following expression:

$$\langle \Psi | H | \Psi \rangle = \langle \Psi | H_0 | \Psi \rangle + \langle \Psi | V | \Psi \rangle$$

$$= (12 \langle \psi_1 | H_0 | \Psi \rangle + 6 \langle \psi_2 | H_0 | \Psi \rangle) + 6 \langle \Psi | V_{12} | \Psi \rangle \tag{59}$$

where

$$\langle \psi_1 | H_0 | \Psi \rangle = 8\pi^2 \int_0^\infty du_1 \, u_1^2 \int_0^\infty du_2 \, u_2^2 \int_0^\infty du_3 \, u_3^2 \, [\frac{u_1^2}{m} + \frac{3u_2^2}{4m} + \frac{2u_3^2}{3m}]$$

$$\times \int_{-1}^{+1} dx_1 \int_{-1}^{+1} dx_2 \int_0^{2\pi} d\varphi_1 \, \psi_1(u_1 \, u_2 \, u_3 \, x_1 \, x_2 \, \varphi_1) \, \Psi(u_1 \, u_2 \, u_3 \, x_1 \, x_2 \, \varphi_1)$$

$$\langle \psi_2 | H_0 | \Psi \rangle = 8\pi^2 \int_0^\infty dv_1 \, v_1^2 \int_0^\infty dv_2 \, v_2^2 \int_0^\infty dv_3 \, v_3^2 \, [\frac{v_1^2}{m} + \frac{v_2^2}{2m} + \frac{v_3^2}{m}]$$

$$\times \int_{-1}^{+1} dX_1 \int_{-1}^{+1} dX_2 \int_0^{2\pi} d\phi_1 \, \psi_2(v_1 \, v_2 \, v_3 \, X_1 \, X_2 \, \phi_1) \, \Psi(v_1 \, v_2 \, v_3 \, X_1 \, X_2 \, \phi_1)$$

$$\tag{60}$$

and

$$\langle \Psi | V_{12} | \Psi \rangle = 8\pi^2 \int_0^\infty du_1 \, u_1^2 \int_{-1}^{+1} dx_1 \int_0^{2\pi} d\varphi_1 \int_0^\infty du_2 \, u_2^2 \int_{-1}^{+1} dx_2 \int_0^\infty du_3 \, u_3^2$$

$$\times \, \Psi(u_1 \, u_2 \, u_3 \, x_1 \, x_2 \, \varphi_1) \int_0^\infty du_1' \, u_1'^2 \int_{-1}^{+1} dx_1' \int_0^{2\pi} d\varphi_1'$$

$$\times \, V_{12}(u_1, u_1', y_{11'}) \, \Psi(u_1' \, u_2 \, u_3 \, x_1' \, x_2 \, \varphi_1') \tag{61}$$

where $y_{11'} = x_1 x_1' + \sqrt{1 - x_1^2} \sqrt{1 - x_1'^2} \cos(\varphi_1 - \varphi_1')$. The expectation values of the kinetic energy $\langle H_0 \rangle$, the two-body interaction $\langle V \rangle$ and the hamiltonian operator $\langle H \rangle$ are given in table 8 for Malfliet-Tjon V potential calculated in 3D scheme. In the same table the four-body binding energy calculated in 3D scheme is also shown for comparison to the expectation values of the Hamiltonian operator. One can see that the energy expectation value and eigenvalues E agree with high accuracy.

VII. SUMMARY AND OUTLOOK

Instead of solving the coupled F-Y equations in a PW basis, we introduce an alternative approach for four-body bound state calculations which implement directly momentum

TABLE VIII: The expectation values of the kinetic energy $\langle H_0 \rangle$, the two-body interaction $\langle V \rangle$ and the Hamiltonian operator $\langle H \rangle$ calculated for Malfliet-Tjon V potential in 3D scheme.

	$\langle H_0 \rangle$ [MeV]	$\langle V \rangle$ [MeV]	$\langle H \rangle$ [MeV]	E [MeV]
FY(3D)	69.7	-101.0	-31.3	-31.3

vector variables. We formulate the coupled F-Y equations for identical spinless particles as function of vector Jacobi momenta, specifically the magnitudes of the momenta and the angles between them. We expect that coupled three-dimensional F-Y equations for a bound state can be handled in a straightforward and numerically reliable fashion. Our results for spin-independent two-body potentials are in good agreement with pervious values for VAR, HHE, SKFR and DFY techniques, especially they are matched with PW calculations in F-Y scheme. Also working directly with momentum vector variables gives the benefit of considering all partial waves, which provides perfect agreement with GFMC, CCE, CR-CGBV, ATMS, VAR, IDEA, DMC, HH, SVM, EIHH and F-Y(PW) values for Malfliet-Tjon V potential. This is very promising and nourishes our hope that calculations with realistic NN potential models, namely considering spin and isospin degrees of freedom, will most likely be more easily implemented than the traditional PW-based method. The stability of our algorithm and our 3D representation of Yakubovsky components have been achieved with the calculation of the eigenvalue of Yakubovsky kernel, where different number of grid pints for Jacobi momenta and angle variables have been used. Also we have calculated the expectation value of the Hamiltonian operator. This test of calculation has been done with Malfliet-Tjon V potential and we have achieved good agreement between the obtained eigenvalue energy and expectation value of the Hamiltonian operator. We predict that the incorporation of three-body forces will most likely also be less cumbersome in a 3D approach.

for guidance to Lapack and Arpack numerical libraries.

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